Trigonometric Proofs for Congruent Triangles

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Abstract

The paper discusses the algebraic/trigonometric approach to prove triangles are congruent. All traditional statements about congruent triangles - SAS, ASA, and SSS - are proved with trigonometric forlmulae. Authors discuss the trigonometric proof of the SSA theorem.

Keywords: congruent triangles; a theorem; a proof; a definition; a deductive proof; an algebraic proof; a trigonometric proof; SAS, ASA, SSS, and SSA conditions; algebraic calculations.

Introduction

It is an internationally accepted point of view that proving triangles are congruent is a well-developed subject in teaching Geometry. Mathematical teachers traditionally use two main approaches in teaching congruency: deductive proofs (reasoning in which a conclusion follows with strict necessity from a premise) and postulates (statements accepted without proofs). However, such approaches have two inherent problems.

The first issue is that often statements about congruent triangles are considered in different text books in different ways. Namely, in some text books, statements about congruent triangles are considered either as theorems or as postulates, while in other text books, the same statements are considered as conjectures or are given without explanation. Very often there are no proofs for theorems, or the proofs are given as problems to be proved by students. Table 1 below shows a brief comparison for statements SAS, ASA, and SSS for congruent triangles in different high school Geometry text books from different countries:

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Book	Author	SAS	ASA	SSS
Geometry	Larson (2007)	postulate	postulate	postulate
Geometry	Burger (2007)	postulate	postulate	postulate
Geometry	Larson (2016)	theorem	theorem	theorem
Plane Geometry	Seymour (2002)	theorem	theorem	theorem
Geometria	Pogorelov (2001)	theorem	theorem	theorem
Geometria	Baldor (2004)	theorem	theorem	theorem
Geometry	Serra (2003)	conjecture	conjecture	conjecture

Table 1. Comparison of statements about congruent triangles in different text books.

Statements about congruent triangles are described differently by different authors. For example, Larson, Boswel and Stiff (2007) consider statements SAS, ASA, and SSS are postulates. At the same time Larson and Boswel (2016) consider all these statements as theorems and also provide proofs. In the book by Edward Burger et al. (2007), statements SAS, ASA, and SSS are postulates. Many other authors, such as Seymour (2012), Pogorelov (2001) or Baldor (2004) all statements about congruent triangles are considered theorems and such proofs are given in that book. On other hand, Serra (2003) considers these statements as conjectures without explanation as to whether they are theorems or postulates.

The second issue in the traditional approach of teaching congruency of triangles is that the SSA statement is considered an invalid approach for obtaining congruency of triangles, and as a theorem leads to possible ambiguity.

There is another paradigm which must be considered in teaching proofs of congruent triangles: the use of a sequence of algebraic and trigonometric calculations. Such an approach seems to be new in teaching congruency of triangles and more effective for students and easier for teachers.

There are articles which describe different methods of deductive proofs for the congruency of triangles. For example, Patkin and Plashkin (2011) describe sufficient and non-sufficient conditions for triangles to be congruent. The traditional deductive proof of the SSA statement was presented by Mironychev (2015). But there is very little literature on the use of algebraic and trigonometric calculations to prove congruency of triangles. Such calculations, as usual, are represented as isolated exercises without generalizations to these calculations as theorems; see for example Lial (2017) or Blitzer (2010).

The purpose of this article is to present the algebraic/trigonometric approach to prove traditional statements of congruent triangles. It turns out that this method can be effectively used for describing the SSA statement for triangle congruency, a statement which traditionally has been considered invalid in proving that triangles are congruent.

If the authors' position is true, then it could be considered as a starting point for the reevaluation of the philosophy of teaching congruent triangles worldwide. First, the statements about congruent triangles (SAS, ASA, SSS, and including SSA) should not be treated differently either as postulates or as theorems or as hypotheses. The fewer the number of postulates a theory has the higher is its scientific level. So, teaching the subject this way will increase the level of education internationally. The matter of the Geometry is common for all countries and should not be treated differently from country to country. The authors hope the new (algebraic/trigonometric) way of proving congruent triangles will influence the development of similar proofs for other topics of Geometry and stimulate book authors and educators from around the world to revise the philosophy of teaching congruent triangles.

Definitions and Proofs of Congruency for Triangles

In general, traditional deductive proofs are based on certain definitions and postulates. Deductive proofs seem to be a very challenging topic for high school students. In *The Elements* (Euclid, 2016, p.247) the proof of the SAS theorem was made without an exact definition of congruency. In *The Elements* congruence is considered an obvious issue and is not required to discuss. One type of definition for congruent figures (triangles) accepted in traditional Geometry courses is that "In two congruent figures (triangles), all the parts of one figure are congruent to the corresponding parts of the other figure. In congruent polygons, this means that the *corresponding sides* and the *corresponding angles* are congruent"(Larsen, 2007, p.225). Such definition, despite its clarity, does not show how to state that sides or angles are congruent.

To make geometric proofs more understandable and affordable for students, text book authors could represent proofs as a sequence of certain algebraic/ trigonometric calculations. In this way it is necessary to accept some new definition of congruency. To make the algebraic or trigonometric calculations the central part of the proof, such a definition has to be related to measures of triangles' parts; namely, lengths of sides and measures of angles. Each triangle has six parts, or elements: three sides and three angles.

So, algebraically, a triangle can be considered as a set of six numbers:

$$< s_1, s_2, s_3, s_4, s_5, s_6 > .$$

Assume that for any three given elements $s_{i_1}, s_{i_2}, s_{i_3}$ (initial data) there are some functional representations (formulae) for the other three elements

$$\begin{aligned} s_{i_4} &= f(s_{i_1}, s_{i_2}, s_{i_3}) \\ s_{i_5} &= \phi(s_{i_1}, s_{i_2}, s_{i_3}) \\ s_{i_6} &= \psi(s_{i_1}, s_{i_2}, s_{i_3}). \end{aligned}$$

So, if such functions $f(\cdot)$, $\phi(\cdot)$, and $\psi(\cdot)$ do exist, then all triangles with the same initial data s_{i_1}, s_{i_2}, and s_{i3} are congruent. From this point of view the definition for the congruency of triangles could be the following:
DEFINITION: If for some triangle $< s_1, s_2, s_3, s_4, s_5, s_6 >$ and given known elements $s_{i_1}, s_{i_2}, s_{i_3}$ there are some unique functions for the other three elements

$$\begin{aligned} s_{i_4} &= f(s_{i_1}, s_{i_2}, s_{i_3}) \\ s_{i_5} &= \phi(s_{i_1}, s_{i_2}, s_{i_3}) \\ s_{i_6} &= \psi(s_{i_1}, s_{i_2}, s_{i_3}), \end{aligned}$$

then all triangles with the same given values $s_{i_1}, s_{i_2}, s_{i_3}$ are congruent.
Consider some triangle $\triangle ABC$ and let us accept the following notations:

$$AB = c,\ BC = a, \text{ and } AC = b.$$

It is possible to state the following correspondence between elements of the triangle: s_1 - is the side a, s_2 - is the side b, s_3 - is the side c, s_4 - is the angle $\angle A$, s_5 - is the angle $\angle B$, and s_6 - is the angle $\angle C$:

$$< s_1, s_2, s_3, s_4, s_5, s_6 >=< a, b, c, \angle A, \angle B, \angle C >.$$

These numbers belong to a certain range. The numbers a, b, and c represent the lengths of triangle sides. So, their range is:

$$a, b, c \in (0, \infty) \text{ or } s_1, s_2, s_3 \in (0, \infty).$$

The remaining three numbers represent angles of a triangle, so their range is:

$$m\angle A, m\angle B, m\angle C \in (0, \pi) \text{ or } s_4, s_5, s_6 \in (0, \pi).$$

If angles are measured in degrees, then the range will be as follows:

$$m\angle A, m\angle B, m\angle C \in (0, 180^o) \text{ or } s_4, s_5, s_6 \in (0, 180^o).$$

SAS Theorem Proved Trigonometrically

The SAS (side-angle-side) statement about congruent triangles states that if two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then such triangles are congruent. Despite the fact that the original statement in Euclid's *Elements* was considered and proved as a theorem, many contemporary text books use this statement as a postulate. Let us prove this statement as a theorem with methods of Trigonometry.

Now assume that sides b, c, and the angle $\angle A$ are given (in other words $< s_2$, s_3, and s_4 are given). The problem is to find expressions for side a, the angle $\angle B$, and the angle $\angle C$ as functions of a, b, and the angle $\angle A$:

$$a = f(b, c, \angle A)$$
$$\angle B = \phi(b, c, \angle A)$$
$$\angle C = \psi(b, c, \angle A)$$

In other words, we must find s_1, s_5, and s_6 as some functions of s_2, s_3, and s_4.

PROOF:

1. Side a can be found from the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$$

From here it follows:

$$a = \sqrt{b^2 + c^2 - 2bc \cdot cos(A)}.$$

Since a is the length of a segment, we know that $a > 0$ and for the square root, only the positive value makes sense.

2. The second angle, for example, the angle $\angle B$, could be either acute or obtuse as represented in the Fig. 1 below. If the angle $\angle B$ is acute (in this case $c > b \cdot cos(A)$), then its value could be found from the Law of Sines:

$$\tfrac{a}{sin(A)} = \tfrac{b}{sin(B)} \ .$$

From here it follows:

$$sin(B) = \frac{b}{a} \cdot sin(A)$$

which allows one to get the expression for the angle $\angle B$:

$$m\angle B = arcsin(\frac{b}{a} \cdot sin(A)). \tag{1}$$

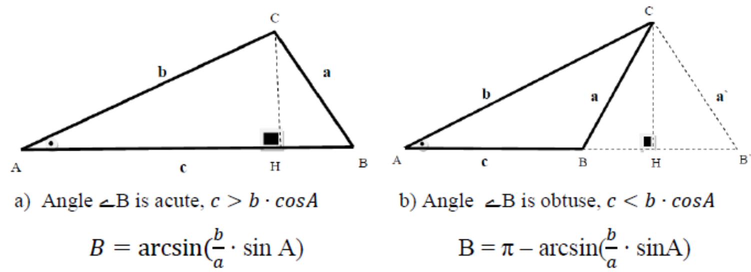

Figure 1: Examples of SAS theorem when a) the angle B is acute and b) when it is obtuse.

If the initial angle $\angle A$ is acute and the angle $\angle B$ is obtuse (when $c < b \cdot cos(A)$), then the supplementary angle for $\angle B$ should be taken from the formula (1):

$$m\angle B = \pi - arcsin(\frac{b}{a} \cdot sin(A)). \tag{2}$$

If $c = b \cdot sin(A)$ then the angle $\angle$B is a right angle:

$$m\angle B = \tfrac{\pi}{2}\ .$$

3. The measure of the third angle $\angle C$ comes from the Interior Angles Theorem:

$$m\angle A + m\angle B + m\angle C = \pi \text{ or } m\angle C = \pi - m\angle A - m\angle B.$$

Summarizing the above calculations, we have the following system of expressions for the missed elements of the triangle $\triangle ABC$:

$$\begin{aligned} a &= \sqrt{b^2 + c^2 - 2bc \cdot cos(A)} \\ m\angle B &= arcsin(\frac{b}{a} \cdot sin(A)) - \text{for acute angles, if } c > b \cdot cos(A) \\ m\angle B &= \pi - arcsin(\frac{b}{a} \cdot sin(A)) - \text{ for obtuse angles, if } c < b \cdot cos(A) \\ m\angle C &= \pi - m\angle A - m\angle B \end{aligned} \tag{3}$$

4. So, all missed elements of the triangle $\triangle ABC$ are represented as algebraic/trigonometric expressions of the initial data - two sides and one included angle. From the algebraic definition of congruency it follows that all tringles with such data are congruent.

END OF THEOREM

Example 1-1.
Given: side $b = 8$ units, side $c = 10$ units, $m\angle A = \pi/6$ or 30^o.
These data show that the angle $\angle B$ is acute. So, from (3) it follows:

$$a = \sqrt{8^2 + 10^2 - 2 \cdot 8 \cdot 10 \cdot cos(30^o)} \approx 5.04 \text{ (units)}$$
$$m\angle B = arcsin(\tfrac{8}{a} \cdot sin(30^o)) \approx 52^o$$
$$m\angle C = 180^o - 30^o - 52^o \approx 98^o.$$

Example 1-2.
Given: side $b = 8$ units, side $c = 6$ units, $m\angle A = \pi/6$ or 30^o.
These data show that the angle $\angle B$ is obtuse. So, from (3) it follows:

$$a = \sqrt{8^2 + 6^2 - 2 \cdot 8 \cdot 6 \cdot cos(30^o)} \approx 4.11 \text{ (units)}$$
$$m\angle B = 180 - arcsin(\tfrac{8}{a} \cdot sin(30^o)) \approx 103^o$$
$$m\angle C = 180^o - 30^o - 103^o \approx 47^o.$$

ASA Theorem Proved Trigonometrically

The ASA (angle-side-angle) statement about congruent triangles states that if two angles and the included side in one triangle are congruent to two corresponding angles and the included side in another triangle, then such triangles are congruent.
Consider a triangle in which we are given two angles and an included side (see the Fig.2).

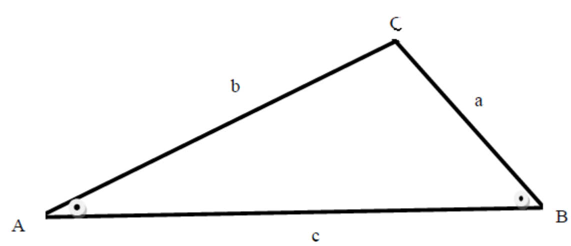

Figure 2: ASA theorem: given are the angle A, the angle B, and the side c.

GIVEN:
$m\angle A$, $m\angle B$, and the length of the side $AB = c$.
PROVE:
Triangles with such initial data are congruent. In other words, it is necessary to find algebraic/trigonometric expressions for other parts of the triangle: the measure of angle $\angle C$ and lengths of sides $AC = b$ and $BC = a$.

$$m\angle C = f(m\angle A, m\angle B, c)$$
$$a = \phi(m\angle A, m\angle B, c)$$
$$b = \psi(m\angle A, m\angle B, c)$$

PROOF:
1. Angle $\angle C$ can be found from the Interior Angle Theorem:

$$m\angle A + m\angle B + m\angle C = \pi \text{ or } m\angle C = \pi - m\angle A - m\angle B.$$

2. For sides a and b we can use the Law of Sines:

$$\tfrac{a}{sin(A)} = \tfrac{b}{sin(B)} = \tfrac{c}{sin(C)} = \tfrac{c}{sin(\pi - m\angle A - m\angle B)} \ .$$

From here it follows:

$$a = \tfrac{c \cdot sin(A)}{sin(\pi - m\angle A - m\angle B)} \text{ and } b = \tfrac{c \cdot sin(B)}{sin(\pi - m\angle A - m\angle B)}.$$

So, expressions for two sides and one angle under considerations are:

$$\begin{aligned} m\angle C &= \pi - m\angle A - m\angle B \\ a &= \frac{c \cdot sin(A)}{sin(\pi - m\angle A - m\angle B)} \\ b &= \frac{c \cdot sin(B)}{sin(\pi - m\angle A - m\angle B)}. \end{aligned} \tag{4}$$

These expressions are unique. All triangles with such initial data are congruent.

END OF THEOREM

Example 2. As an application of this theorem consider a triangle $\triangle ABC$.
Given: Angle $m\angle A = \pi/6$ (30^o), angle $\angle B = \pi/4$ (45^o), and the length of the side $AB = c = 10$ units.
Find: Side $AC = b$, side $BC = a$, and the angle $\angle C$.
Solution: In accordance with (4) above it follows that:

$m\angle C = \pi - \pi/6 - \pi/4 = 7\pi/12$ (105^o)
$a = \frac{10 \cdot sin(\pi/6)}{sin(\pi - \pi/6 - \pi/4)} \approx 5.2$ (units)
$b = \frac{10 \cdot sin(\pi/4)}{sin(\pi - \pi/6 - \pi/4)} \approx 7.3$ (units).

SSS Theorem Proved Trigonometrically

The SSS (side-side-side) statement about congruent triangles states that if three sides in one triangle are congruent to three corresponding sides in another triangle, then such triangles are congruent.
Consider a triangle $\triangle ABC$ with three given sides $AB = c$, $BC = a$, and $AC = b$ (see the Fig.2 above). In this case it is necessary to find measures of angles $\angle A$, $\angle B$, and $\angle C$.
GIVEN: Lengths of sides a, b, and c.
PROVE: Tringles with such initial data are congruent. In other words, it is necessary to find algebraic/trigonometric expressions for measures of the angles $\angle A$, $\angle B$, and $\angle C$:

$$m\angle A = f(a, b, c)$$
$$m\angle B = \phi(a, b, c)$$
$$m\angle C = \psi(a, b, c)$$

PROOF: Functions $f(\cdot), \phi(\cdot)$ and $\psi(\cdot)$ could be found on the basis of the Law of Cosines and the Interior Angles Theorem in the following way.
1. From the Law of Cosines for the angle $\angle A$:

$$a^2 = b^2 + c^2 - 2ab \cdot cos(A)$$

it follows:

$$m\angle A = arccos(\tfrac{b^2+c^2-a^2}{2bc}).$$

2. From the Law of Cosine for the angle $\angle B$:

$$b^2 = a^2 + c^2 - 2ac \cdot cos(B)$$

it follows:

$$m\angle B = arccos(\tfrac{a^2+c^2-b^2}{2ac}).$$

Once angle $\angle A$ and angle $\angle B$ are known, the third angle $\angle C$ comes from the Interior Angles Theorem:

$$m\angle C = \pi - m\angle A - m\angle B.$$

So, expressions for the angles under consideration are:

$$\begin{aligned} m\angle A &= arccos(\frac{b^2 + c^2 - a^2}{2bc}) \\ m\angle B &= arccos(\frac{a^2 + c^2 - b^2}{2ac}) \\ m\angle C &= \pi - m\angle A - m\angle B. \end{aligned} \tag{5}$$

All these expressions are unique. All triangles with such initial data are congruent.

END OF THEOREM

NOTE. The second angle, $\angle B$, could also be found on the basis of the Law of Sines rather than the Law of Cosines as it was done above.

Example 3.
As an application of this theorem consider a triangle $\triangle ABC$ (Fig. 3) with the following data:
Given: side $a = 7$ units, side $b = 5$ units, and side $c = 9$ units.
Find: angle $\angle A$, angle $\angle B$, and angle $\angle C$.

Solution: From above formulae it follows:

$$m\angle A = arccos(\frac{b^2+c^2-a^2}{2ab}) = arccos(\frac{5^2+9^2-7^2}{2\cdot 5\cdot 9}) \approx 51^0,$$
$$m\angle B = arccos(\frac{a^2+c^2-b^2}{2ac}) = arccos(\frac{7^2+9^2-5^2}{2\cdot 7\cdot 9}) \approx 35^0,$$
$$m\angle C = 180^0 - 51^0 - 35^0 \approx 94^0.$$

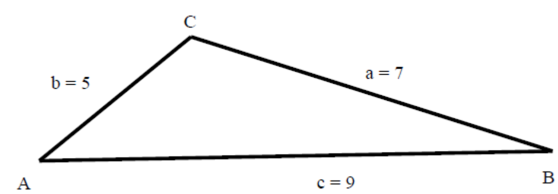

Figure 3: SSS theorem: given are sides a, b, and c.

So, for all triangles with the given sides $a = 7$ units, side $b = 5$ units, and side $c = 9$ units the measures of angles $\angle A$, $\angle B$, and $\angle C$ will be the same. Such triangles are congruent.

SSA Theorem Proved Trigonometrically

Among statements about congruent triangles, the SSA statement requires special consideration. SSA (side-side-angle) statement about congruent triangles states that triangles with two congruent sides and one congruent angle which is not between these congruent sides **could** be congruent if the corresponding angles are acute or obtuse. The traditional deductive proof

of this theorem was done by Mironychev (2015). This theorem could be proved trigonometrically as well.
Consider, for example, that sides a, b, and the angle $\angle A$ are given and the angle $\angle A$ is not between sides a and b. In this case the following options are possible (Fig. 3-5):

- If $b \cdot sin(A) > a$, then no triangles exist under this condition and as such, there is no solution;
- If $b \cdot sin(A) = a$ or $a > b$, then there is one solution;
- If $b \cdot sin(A) < a < b$, then there are two possible solutions.

Let us go through all these options.

1. If $b \cdot sin(A) > a$, then such data cannot form a triangle. This case is represented in the Fig.3 below.

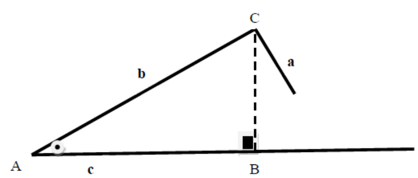

Figure 4: SSA theorem: the angle A and sides b and c cannot form a triangle.

2. If $b \cdot sin(A) = a$ then $m\angle B = 90^o$, side $c = a \cdot cos(A)$, and $m\angle C = \pi - m\angle A - m\angle B$. This a regular case for right triangles.

   If $a > b$, then angle $\angle$B is acute. This case is represented in the Fig.4 below. Angle $\angle B$ could be found from the Law of Sines:

$$\frac{a}{sin(A)} = \frac{b}{sin(B)}. \tag{6}$$

   From (6) it follows that the expression for the angle $\angle$B is:

$$m\angle B = arcsin(\tfrac{b}{a} \cdot sinA).$$

   Once $\angle$A and $\angle$B are known, the expression for $\angle$C comes from the Interior Angles Theorem:

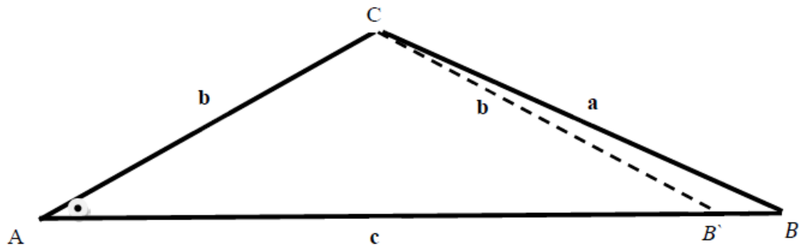

Figure 5: SSA theorem: If $b < a$ then the angle B is acute and there is one solution.

$$m\angle C = \pi - m\angle A - m\angle B.$$

The expression for the side c can be found from the Law of Cosines:

$$c = \sqrt{a^2 + b^2 - 2ab \cdot cos(C)}.$$

3. If $b \cdot sin(A) < a < b$, then for the same a there are two solutions depending on whether the angle $\angle$B is acute or obtuse in triangles $\Delta AB`C$ or $\Delta AB``C$ as represented in the Fig.5 below.
   If the angle $\angle$B is acute ($\Delta AB```C$), then as in the case 2 above the

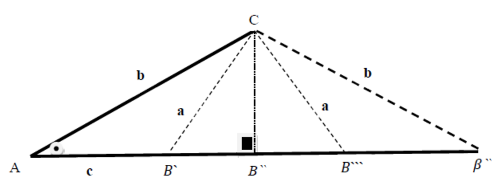

Figure 6: SSA theorem: If $a < b$ then there are two triangles with same side a, side b and the angle A.

expression for the angle $\angle B$ would be:

$$m\angle B = arcsin(\frac{b}{a} \cdot sinA).$$

If $\angle$B is obtuse ($\Delta AB`C$), then the expression for $\angle B$ would be:

$$m\angle B = \pi - arcsin(\frac{b}{a} \cdot sinA).$$

From here $\angle$C comes from the Interior Angles Theorem and the side c comes from the Law of Cosines in the same way as above:

$$m\angle C = \pi - m\angle A - m\angle B,$$
and
$$c = \sqrt{a^2 + b^2 - 2ab \cdot cos(C)}.$$

So, for the given sides a, b, and the angle ∠A there exist unique expressions for ∠B, ∠C, and side c. All triangles with such initial data are congruent in accordance with the algebraic definition of congruency.

END OF THEOREM

Example 4.
Given: Side $b = 8$ units, side $a = 5$ units, angle $\angle A = 30^o$, angle $\angle B$ is acute, and angle $\angle C$ is obtuse.
Find: Measures of angle $\angle B$, $\angle C$, and the length of the side c.
Solution: From above it follows:

$m\angle B = arcsin(\frac{b}{a} \cdot sin(A)) = arcsin(\frac{5}{8} \cdot sin(30^o)) \approx 53^o$
$\angle C = 180^o - 30^o - 53^o \approx 97^o$
$c = \sqrt{a^2 + b^2 - 2ab \cdot cos(C)} = \sqrt{5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot cos(C)} \approx 9.94$ (units).

So, all missed elements of the triangle $\triangle ABC$ are represented as algebraic/trigonometric functions of the initial data. From the algebraic definition of congruency, it follows that all triangles with such initial data are congruent.

Discussion of Results

In this article, the authors have proved the congruency of triangles using a sequence of algebraic and trigonometric computations rather than the traditional deductive approach used by teachers internationally. Such style of proof seems easier for the contemporary generation of students, as this approach is more aligned with a computational style of thinking rather than traditional deductive thinking. In the traditional approach using deductive reasoning, students must use postulates and previously proved theorems to make a conclusion about congruency of triangles. However, with the algebraic/trigonometric approach, students need only produce a sequence of calculations from given data to make conclusions about the congruency of triangles. For example, one need only input the initial data necessary to satisfy the hypothesis of the theorem, and through the calculations described in the proof, the remaining data points are generated. As a result, these proofs can be easily adapted to help the student make real-world connections and solve real-life applications; the deductive reasoning approach of proofs leaves no room for students to compute actual numerical values.

Any mathematical inquiry must begin with a clear understanding of the object of investigation and its intrinsic or internal properties. Therefore, the authors have given a new definition of congruent triangles, a definition which is based on algebraic function values. The advantage of using this new definition is that the teacher and student are given the opportunity to apply algebraic and trigonometric methods for proving triangles are congruent.

Additionally, the authors have proved the SSA statement as a theorem. What is significant about this theorem is that traditionally, any proof which relies on the SSA statement is immediately regarded as an invalid proof because of the possibility of the ambiguity. When initial data satisfies the SSA statement (in other words, what is given are lengths of two sides of a triangle and the measure of the angle that is not between these sides), it is possible that either no such triangle exists, one triangle exists, or two such triangles exist, creating ambiguity. However, the authors have shown that the SSA statement can be considered a true theorem. The example (Benson et al (2012), p. 156) states that the two triangles presented are not congruent because of SSA, although these triangles are indeed congruent. Such immediate dismissal of the SSA statement of proving triangles congruent must be revised by educators and publishers alike (see the Picture 7). More similar examples can be found in Mironychev (2015).

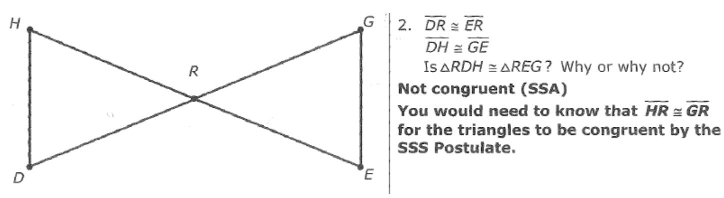

Figure 7: Problem No 2 on page 156 from Benson et al. (2012).

Finally, the authors believe that trigonometric proofs for congruent triangles can be included in Trigonometry courses, linking trigonometry and geometry. In traditional trigonometry courses such as Lial (2017) or Blitzer (2010), students are asked to solve a triangle, which means that students must find the missing pieces of a triangle, given some initial information (data) about the triangle. The authors recognize that the topic of solving triangles should be considered as theorems.

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References

Baldor, J.A. (2004). *Geometria Plana Y Del Espacio*. Mexico: Publicaciones Cultural.

Blitzer, R. (2010). *Precalculus*. New Jersey: Pearson.

Burger, E.B., Chard, D.J., Hall, E.J., Kennedyd, P.A., Leinward, S.J., Renfro, F.L., & Waits, B.K. (2007). *Geometry*. Orlando: Holt.

Euclid (2016). *The Elements*, vol.1. New York: Dover.

Larsen, R., Boswel, L. & Stiff, L. (2007). *Geometry*. New York: McDougal Littell.

Larsen, R., Boswel, L. (2016). *Geometry*. Erie, PA: Big Ideas Learning.

Lial, M. (2017). *Trigonometry*. American River College: Pearson.

Mironychev, A. (2015). Congruency Theorems in the Geometry Curriculum of High Schools. Proceedings of the conference *Universality of Global Education Issues*, vol.2, Sam Houston State University. Retrieved from http://digital.library.shsu.edu/cdm/compoundobject/ collection/p16042coll4/id/56.

Patkin, D., and Plashkin, O. (2011). Congruent Triangles. Sufficient and insufficient conditions. Suggested Milestones for Inquiry and Discussion. *Journal of Korean Educational Series D: Research in Mathematical Education*, vol.10, No 4, pp. 327-340.

Pogorelov, A.V. (2001). *Geometria*. Moscow: Prosveshenie.

Serra, M. (2003). *Geometry*. Emerville: Key Curriculum Pres.

Seymour, F.E. (2002). *Plane Geometry*. Pensacola, FL: A Beka Book.

Benson, Sh., Henry, Y., Horn, J., Nicodemo, P., Riggs, R. & Seymour, K. (2012). *Supporting STAAR Achievement: Targeting the TEKS and Readiness Standards: Geometry*. Houston, TX: Region 4 Education Service Center.