Congruency Theorems in the Geometry Curriculum of High Schools: An International Comparison

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Abstract

This study investigates different approaches in contemporary math education. The author compares his experience in teaching Geometry in Russia and in the United States. The roles of Math and its purposes in the high school curriculum are discussed. The focus is made on theorems about congruent triangles. The set of congruency theorems can be arranged in different ways depending on the level of education and diversity of students.

The central theme of the study is an investigation of the theorems and postulates used to conclude triangles are congruent. Every triangle consists of three angles and three sides – a total of six elements. For triangles to be congruent all these elements must be congruent. Specific theorems require only three elements to be congruent to determine the congruency of triangles. These theorems are: SAS theorem (it compares two sides and one included angle), ASA and AAS theorems (they compare two angles and one side), SSS theorem (it compares three sides), HL theorem (a specific case for right triangles). The problem under consideration is the case of SAS theorem in which the congruent angles are not the included angles between congruent sides. The author describes different cases related to this issue and investigates conditions regarding the congruency of such triangles – SSA conditions. Finally, proposals are made about options to teach congruence theorems for regular high school mathematics classes as well as Pre-AP mathematics classes.

Keywords: math education, Geometry, postulates, theorems, congruent triangles, proofs of theorems, SAS and SSA conditions.

Preface

The purpose of this article is to compare aspects of teaching high school Geometry on the basis of the author's experiences in Houston, Texas (at the Houston Independent School District (HISD)) and in Moscow (Russia). The first aim is to compare the Geometry curricula, then to compare statements about congruent triangles, and finally to discuss the statement about congruent triangles when the congruent angle is not the included angle between congruent sides. In conclusion the author describes some recommendations regarding the teaching Geometry and proven theorems.

Geometry in High School Curriculum

Math is one of the main core subjects in the high school education in all developed and developing countries. In Texas, the Texas Education Agency (TEA) requires each high school student to obtain 4 credits in Math – Algebra I, Geometry, Algebra II, and one elective course which could be Pre-Calculus, Statistics, Advanced Quantitative Reasoning (AQR), or Math Models.

The aim of Geometry in the school curriculum is not only to acquaint students with basic facts about figures around us but also (and what might be the most important) to develop thinking skills, ability to analyze problems and do generalizations.

Table 1. Skills developed by Geometry.

Math Related Skills	Non- Math Related Skills
1. Describe properties of Geometric Figures 2. Represent geometric properties algebraically 3. Derive formulae for different measures – Length, Area, Volume 4. Describe Congruence and Similarity	1. Abstract thinking 2. Logic skills 3. Make classifications 4. Determine relationships 5. Do proofs 6. Make conclusions

The two columns below compare briefly the Geometry curriculum in Texas and in Russia as well as the length of time for teaching the scope of this curriculum. The scope and sequence for USA is based on the TEKS (Texas Essential Knowledge and

Skills) – standards accepted for Texas education (TEKS, 2014). The scope and sequence for Russia is based on the Standards of Education of the Russian Ministry of Education (Rossiysky Obsheobrazovatelny Portal, 2015).

USA	Russia
Students learn Geometry one year. The teaching of Geometry could be arranged either in the one block schedule (every day with 45 minute lessons) or in the double block schedule (every other day with 90 minute lessons). For the school year, the total time spent is approximately 128 hours.	In Russia students study Geometry in conjunction with Algebra for 5 years, from 7th grade to 11th grade. Students have 2 Geometry lessons a week and each lesson lasts 45 minutes for a total of 68 (academic) hours each year, or 340 hours for the five-year duration of the subject
1st Cycle (August – October) Students learn main terms of Geometry, representation of points and lines on the coordinate plane, distances and midpoints, and parallel and perpendicular lines. Students also begin learning congruent triangles.	In the **7 grade** (Middle school) Students learn main terms of Geometry; properties of lengths, angles and triangles; relationships in triangles; and properties of parallel lines. Students also learn drawings of figures.
2nd Cycle (October – November) Students learn relationships in triangles (mid-segments, points of concurrency, inscribed and circumscribed circles) transformations of figures, and properties of quadrilaterals and polygons.	In the **8th grade** (Middle school) students learn properties of parallelograms and quadrilaterals, area of figures, Pythagorean theorem, properties of similar figures, basics of trigonometry, geometric transformations of figures, equations of circles on the coordinate plane, and basics of vectors.
3rd Cycle (November – December) Students learn Quadrilaterals and Polygons – properties of parallelograms, kites, and trapezoids. In this term students also learn the Pythagorean theorem and properties of Right Triangles.	The **9th grade** is devoted to trigonometric ratios, the scalar product of vectors, inscribed and circumscribed polygons, the areas of figures, trigonometric ratios, and circumference and area of the circle.
4th Cycle (January – February) Students continue to learn properties of Right Triangles, including Trigonometric Ratios. Students also learn Areas of Figures.	In the **10th grade** students lean basics of stereometry (3-dimensional Geometry) which includes mutual position of lines and planes in space, the intersection of lines

5th Cycle (February – April) Students begin to learn 3D Geometry, including properties of Polyhedrons – surface areas and volumes.

6th Cycle (April – May) Students learn properties of circles, length and area of sectors.

and planes, angles between planes and lines, vectors in three-dimensional space, and properties of polyhedra.

In **11th grade** the main topics are the properties of symmetry, transformation of figures with motion, and volumes of 3D figures.

Comparison of Geometry Text books

Textbooks are considered a key tool when studying a subject. Students are required to use books during the lesson and read books at home when doing home assignments. Below is a comparison of typical American and Russian books by physical size and content. Despite the numerous differences, both books have advantages and disadvantages, depending upon the educational purposes, the students' needs, and the availability of time for the course.

American textbooks

Textbook (Larson, R., Boswell, L., Kanold, T.D., Stiff, L., 2007) is about $10\frac{3}{4} \times 8\frac{1}{2} \times 1\frac{1}{2}$ inches, weighs about 5 pounds.

The book contains more than 1000 pages with several thousands of pictures from real life.

Along with the Geometry content, the textbook contains many facts and topics from other fields of Math such as Arithmetic, Algebra I, and basic Statistics.

The majority of the theorems are not proved in the textbook. Many proofs are suggested to be proved as exercises.

Russian textbooks

Textbooks (Pogorelov, 2014), are rather small, with the common size about $8\frac{1}{2} \times 5\frac{3}{4} \times \frac{3}{4}$ inches with 400 pages. Its weight reaches only ¾ pound.

Textbook has 380 pages and 370 pictures related mainly to geometric figures.

Textbooks have no reviews from Algebra or Arithmetic. The knowledge of these courses in considered as prerequisites for the Geometry course.

All theorems are proved in the textbooks.

It is a modern trend in contemporary education to substitute proofs of theorems with *demonstrations* of computer animation or *hands-on activities* with numerical examples. Another new trend in contemporary education is the transition from paper books to electronic books. Public schools have begun providing students with laptops

with embedded textbooks on all subjects. Such policy eliminates the need for students to carry heavy paper-bound books.

Triangle Congruency Statements

In Geometry all statements about figures and their properties can be arranged in the following way – main terms, definitions, postulates, theorems, lemmas, and corollaries. *Main terms* describe basic figures and are considered to have no exact definition. *Definitions* describe objects on the basis on main terms. *Postulates* state properties of main terms and are accepted without proofs. *Theorems* represent statements regarding properties of figures and require proofs. *Lemmas* are considered as auxiliary theorems which are important for proofs of a more "serious" nature. *Corollaries* are conclusions from theorems. The set of all these statements in a Geometry course depends substantially upon the philosophy of the educational system as well as the goals of that course. In some courses, for simplicity and brevity, certain theorems could be treated as postulates; definitions could be stated in different ways, and in such a cases, the set of theorems would be different for that particular course[1].

The two columns in the Table 2 below compare the definitions of congruency and statements about congruent triangles in traditional American and Russian textbooks. The references reflect the common trend in teaching the concept as well as statements about congruent triangles.

Table 2. Congruency statements.

Congruency statements in American textbooks	Congruency statements in Russian textbooks
In the American Geometry courses the term "congruent figures" is defined as *figures with the same size and shape* (Larson et all, 2007, p. 225). An alternative definition states that in congruent figures corresponding sides and corresponding angles are congruent.	In the Russian Geometry courses the term "congruent figures" is defined as *figures which will coincide after we match them (mentally or physically)* (Atanasian, L.S., Butuzov, V.F., Kadomzev S.B., 2015, p.28). From such a definition it follows that congruent figures must have congruent

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[1] For example, a parallelogram can be defined as *a quadrilateral with opposite parallel sides.* From such a definition, it is easy to prove the theorem that opposite sides are congruent. On the other hand, a parallelogram might be defined as *a quadrilateral with opposite congruent sides.* From this definition the theorem can be proved that opposite sides must be parallel.

All known statements (5 statements) about congruent triangles are learned in the following sequence:

- SSS postulate
- SAS postulate
- HL theorem
- ASA postulate
- AAS theorem (which is derived from the ASA postulate but is not explicitly proved in the textbook)

sides and congruent angles.

In Russian courses all statements about congruent triangles are treated as theorems. Theorems are learned in the following sequence:

- SAS theorem
- ASA theorem
- SSS theorem
- HL theorem

The AAS theorem is considered as a corollary to the ASA theorem.

Many American textbooks (even college-level textbooks) do not include explicit proofs, opting instead to leave many proofs as exercises for students. Not many proofs can be found in the Appendix at the end of the book. Russian textbooks traditionally provide proofs of all theorems mentioned in the course. Moreover, the teacher proves these theorems in class and students repeat proofs as homework or as a classwork.

The use of computer programs and software such as a Sketchpad (Sutherland, 1980), allows the teacher to provide numerical and visual examples to illustrate the validity of statements of the proofs they teach. Often such demonstrations substitute the exact proofs.

Theorem about two congruent sides and one congruent angle

One of the four (or five – see the table above) statements about congruent triangles is the *SAS* statement which says that *two triangles are congruent if two sides and the included angle in one triangle are congruent to two corresponding sides and the included angle in another triangle*. The figures on the Picture1 below demonstrate this statement.
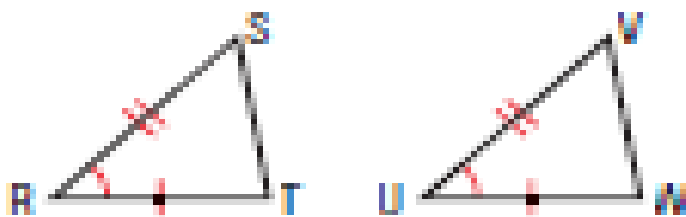
Picture 1. SAS (side-angle-side) statement about congruent triangles: ΔRST ≅ ΔUVN

In triangles ΔRST and ΔUVN, the pairs of sides $\overline{RS}$, $\overline{UV}$ and $\overline{RT}$, $\overline{UN}$ are congruent. The included angles (between congruent sides), $\angle R$ and $\angle U$, are also congruent. So, triangles ΔRST and ΔUVN must be congruent: ΔRST ≅ ΔUVN.

In American tradition, SAS statement is considered a postulate and is accepted as such in many American textbooks. In Russian (and European) tradition, the SAS statement is considered a theorem which is to be proved. The central point of the SAS theorem (postulate) is the proposition that two triangles are congruent if they have *two congruent sides and congruent included angles.* In such a case the triangles are certainly congruent – it is either accepted as a postulate (American approach), or proved as a theorem (European or Russian approach). However, what if the congruent angle *is not* the included angle between congruent sides? In such a case, the triangles under consideration could be different (not congruent). Such a statement often is considered as a SSA statement. A close analysis of the SSA statement shows that triangles that satisfy this condition need not be congruent.

The Picture 2 below demonstrates the case in which two triangles with two congruent sides and congruent non-included angles are not congruent.
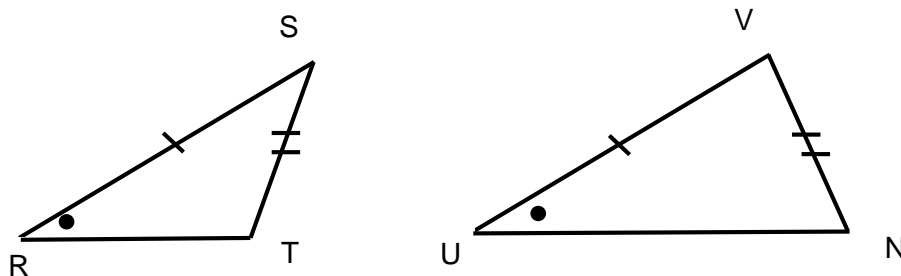
Picture 2. Example when triangles are not congruent if the congruent angles are not included angles for congruent sides.

Pairs of sides $\overline{RS}$, $\overline{UV}$ and $\overline{ST}$, $\overline{VN}$ are congruent but congruent angles $\angle R$ and $\angle U$ are not between the congruent sides. It is clear that these two triangles ΔRST and ΔUVN are not congruent because they have different shapes - angles $\angle S$ and $\angle V$ are

different ($\angle S$ is acute and angle $\angle V$ is obtuse), angles $\angle T$ and $\angle N$ are different ($\angle T$ is obtuse and $\angle N$ is acute), and sides $\overline{RT}$ and $\overline{UN}$ are different (side $\overline{RT}$ is shorter than the side $\overline{UN}$). So, the SSA condition does not provide the congruency for these triangles.

Now consider the question – if for two triangles their congruent angles are not the included angles between congruent sides then *could* such triangles be congruent?

For example, the two triangles pictured below (see the Picture 3) have two pairs of congruent sides and a pair of congruent angles which are not the included angles for these given sides.
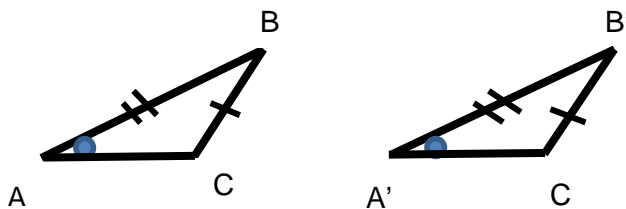
Picture 3. Example when congruent angles are not included angles for congruent sides.

In this picture congruent angles $\angle A$ and $\angle A'$ are not the included angles for pars of congruent sides $\overline{AB}$, $\overline{BC}$ and $\overline{A'B'}$, $\overline{B'C'}$. Corresponding pair of angles $\angle B$, $\angle B'$ are acute, angles $\angle C$, $\angle C'$ are obtuse, but their exact measures are unknown. The question we must consider – are triangles ΔABC and ΔA'B'C' congruent?

Some authors investigate cases in which the triangles described above are certainly congruent as well as cases in which the triangles are not necessarily congruent (Congruence, 2014). They compare the lengths of the opposite and adjacent sides of the congruent non-included angle of each triangle. They argue that if the side opposite the congruent angle is longer than the adjacent side then such triangles are certainly congruent. In the web site of the Oswego School District the statement SSA was named as "donkey theorem" (Regents Exam Preparation Center, 2012). Authors investigate this statement and conclude from some examples that such theorem is not always true. The problem of the "fourth" statement about congruent triangles was discussed at Moscow State University (Rozov, 2010). He described the drawing of different possible triangles with the given congruent angle and two sides. He demonstrated that triangles which satisfy SSA condition are not necessarily congruent and could be different. In all these works authors leave the open question – describe all possible cases when triangles which satisfy SSA conditions are congruent.

So, to answer the question whether triangles that satisfy SSA condition are congruent, consider all possible combination of such triangles - triangles with two congruent sides and one congruent angle which is not an included angle between these sides (SSA condition).

Angles in triangles could be either acute or obtuse. In this article we do not consider right triangles. In congruent triangles all corresponding angles have the same measure. So, if one angle is acute, then the corresponding angle in another triangle has to be also acute. And if some angle is obtuse, then its corresponding angle has to be obtuse as well. So, for any angle there are 2 possible options. But once the sum of measures in the triangle equals 180˚, then for two triangle there are 6 combinations for angles. To answer the question whether triangles that satisfy SSA conditions are congruent, it is necessary to analyze all these six combinations of triangles and angles (the Table 3 below).

It turns out that from all possible 6 cases, there are only two cases (5 and 6) when triangles certainly cannot be congruent as they have different angles. In cases 1-4 triangles have the same similar corresponding angles (acute or obtuse), but the exact measure stays unknown. Such triangles must be congruent.

Table 3. Combinations of angles that make two triangles be congruent or not congruent.

	Non included Congruent angles	Second included angles (exact measure is unknown)	Third included angles (exact measure is unknown)	Congruent triangles
1	Acute	Obtuse	Acute	YES
2	Acute	Acute	Acute	YES
3	Acute	Acute	Obtuse	YES
4	Obtuse	Acute	Acute	YES
5	Acute	Obtuse / Acute	Acute / Obtuse	NO
6	Acute	Acute / Obtuse	Obtuse / Acute	NO

The analysis below proves that in cases 1-4, triangles MUST to be congruent. Proofs for all these cases are quite similar. There are two cornerstones in such proofs. The first issue is that the property of isosceles triangles - their base angles are acute and congruent. This issue is beyond this article to prove. It could be found in any Geometry text book (Larsen et all, 2007), (Burger et all, 2007).

The second issue is the way to compare triangles. Segments are considered congruent if they have the same length (Larsen et all, 2007, p.11). In other words, congruent segments will coincide in all points if we put them on each other. Angles are

considered congruent if they have the same measure (Larsen et all., 2007, p.26). In other words, rays of angles will coincide in all points if we put angles on each other. So, for triangles to be congruent their sides and angles have to be congruent (Larsen et all., 2007, p.225). In other words, congruent triangles coincide in all points if we put them on each other. Such approach will be used for proving statements below.

1. Proofs of Statements

Below are proofs for statements 1-4 from the table 2 above.

Statement 1. If two triangles satisfy SSA conditions, included angles are obtuse, and the third angles are both acute, then such triangles are congruent.

Consider two triangles ΔABC and ΔA'B'C' (see Picture 4 below).
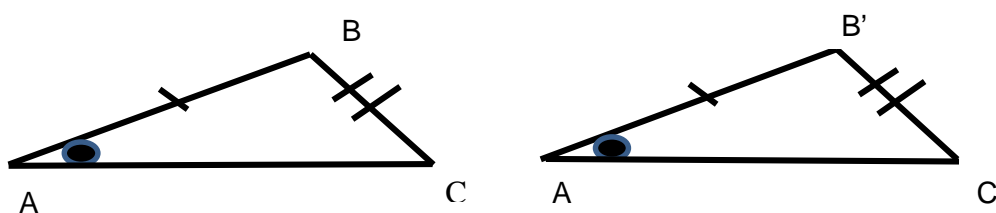
Picture 4. Triangles which satisfy SSA conditions in the Statement 1.

Given: m∠A =   m∠A', $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$; angles ∠B and ∠B' are obtuse; angles ∠C and ∠C' are acute.

Prove: ΔABC ≅ ΔA'B'C' (triangles ΔABC and ΔA'B'C' are congruent).

Proof:

1.   Put the triangle ΔA'B'C' on the triangle ΔABC (see Picture 5) and suppose that these triangles do not coincide. In other words, they are not congruent.
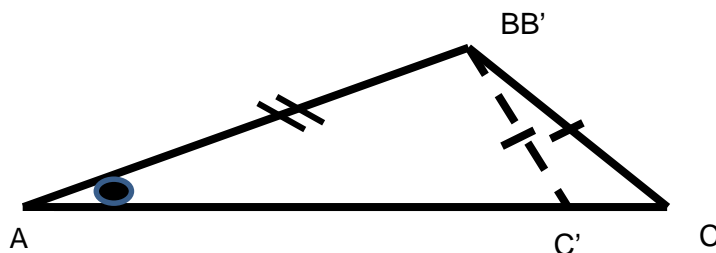
Picture 5. Triangles after matching in the Statement 1.

2. Side $\overline{A'C'}$ will go along side $\overline{AC}$

3. Side $\overline{A'B'}$ will go along the side $\overline{AB}$ and will coincide with $\overline{A'B'}$ because m∠A = m∠A' and $\overline{AB} = \overline{A'B'}$.
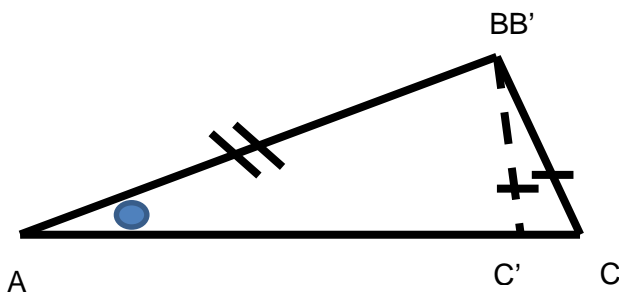
4. Sides BC and B'C' will make an isosceles triangle ΔC'BB'C because $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$. In such case ∠CC'BB' has to be acute base angle in the isosceles triangle. But this is impossible because ∠A'C'B' is acute, angles ∠A'C'B and ∠BB'C'C are supplementary, and that is why ∠CC'BB' turns out to be obtuse.

So, triangles ΔABC and ΔA'B'C' must coincide in all sides, in other words they are congruent.

End of the proof.

Statement 2. If two triangles satisfy SSA conditions, included angles are acute, and the third angles are also both acute, then such triangles are congruent.

Consider two triangles ΔABC and ΔA'B'C' (see Picture 6 below).
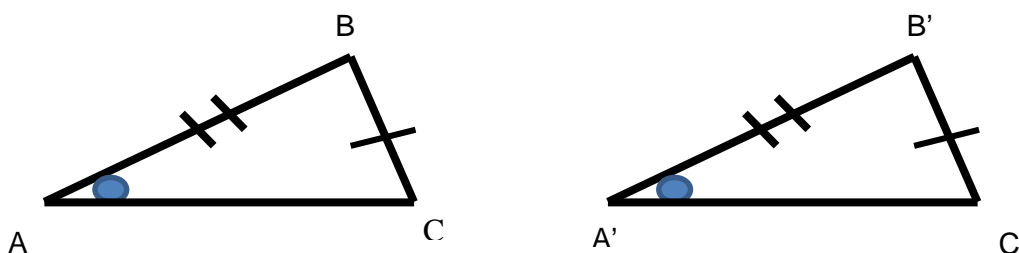
Picture 6. Triangles that satisfy SSA conditions in the Statement 2.

Given: m∠A = : m∠A'; $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$; angles ∠B and ∠B' are acute; angles ∠C and ∠C' are also acute.

Prove: ΔABC ≅ ΔA'B'C' (triangles ΔABC and ΔA'B'C' are congruent).

Proof: 1. Put the triangle ΔA'B'C' on the triangle ΔABC (see Picture 7) and suppose that these triangles do not coincide. In other words, they are not congruent.

Picture 7. Triangles after matching in the Statement 2.

2. Side $\overline{A'C'}$ goes along side $\overline{AC}$

3. Side $\overline{A'B'}$ goes along side $\overline{AB}$ and coincides with $\overline{A'B'}$ as m∠A = m∠A' and ; $\overline{AB}$ = $\overline{A'B'}$.

4. Sides $\overline{BC}$ and $\overline{B'C'}$ make an isosceles triangle ΔC'BB'C as $\overline{AB}$ = $\overline{A'B'}$ and $\overline{BC}$ = $\overline{B'C'}$. In such case ∠CC'BB' has to be acute as a base angle in the isosceles triangle. But this is impossible because ∠A'C'B' is acute, angles ∠A'C'B and ∠BB'C'C are supplementary, and ∠CC'BB' turns out be obtuse.

So, triangles ΔABC and ΔA'B'C' have to coincide in all sides. In other words they are congruent.

End of the proof.

Statement 3. If two triangles satisfy SSA conditions, included angles are acute, and the third angles are both obtuse, then such triangles are congruent.

Consider two triangles ΔABC and ΔA'B'C' (see Picture 8 below).

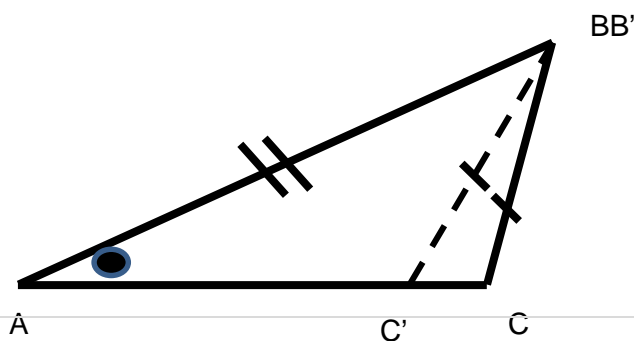Picture 8. `Triangles that satisfy SSA conditions in the Statement 3.

Given: m∠A = : m∠A'; $\overline{AB}$ = $\overline{A'B'}$, and $\overline{BC}$ = $\overline{B'C'}$; angles ∠B and ∠B' are acute; angles ∠C and ∠C' are obtuse.

Prove: ΔABC ≅ΔA'B'C' (triangles ΔABC and ΔA'B'C' are congruent).

Proof:

1. Put the triangle ΔA'B'C' on the triangle ΔABC (see Picture 9) and suppose that these triangles do not coincide. In other words, they are not congruent.

Picture 9. Triangles after matching in the Statement 3.

2. Side $\overline{A'C'}$ goes along side $\overline{AC}$

3. Side $\overline{A'B'}$ goes along side $\overline{AB}$ and coincides with $\overline{A'B'}$ because   m∠A = m∠A' and $\overline{AB} = \overline{A'B'}$.

4. Sides $\overline{BC}$ and $\overline{B'C'}$ make an isosceles triangle ΔC'BB'C because $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$. In such case ∠C has to be acute. But this is impossible because ∠C is an obtuse angle. So, in the isosceles triangle ΔC'BB'C one base angle is acute and the other one is obtuse, which is impossible.

So, triangles ΔABC and ΔA'B'C' must coincide in all sides, so they are congruent.

End of the proof.

Statement 4. If two triangles satisfy SSA conditions, included angles are acute, and the third angles are both obtuse, then such triangles are congruent.

Consider two triangles ΔABC and ΔA'B'C' (see Picture 10 below).
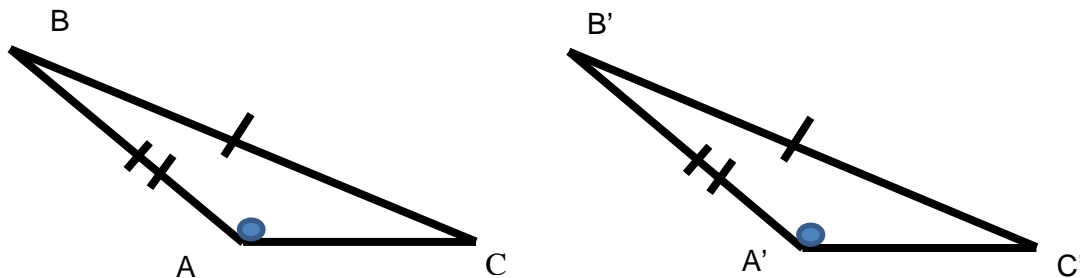
Picture 10. Triangles that satisfy SSA conditions in the statement 4.

Given: m∠A = : m∠A'; $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$; angles ∠B and ∠B' are acute; angles ∠C and ∠C' are acute.

Prove: ΔABC ≅ ΔA'B'C' (triangles ΔABC and ΔA'B'C' are congruent).

Proof:

1. Put the triangle ΔA'B'C' on the triangle ΔABC (see Picture 11) and suppose these triangles do not coincide, in other words, they are not congruent.
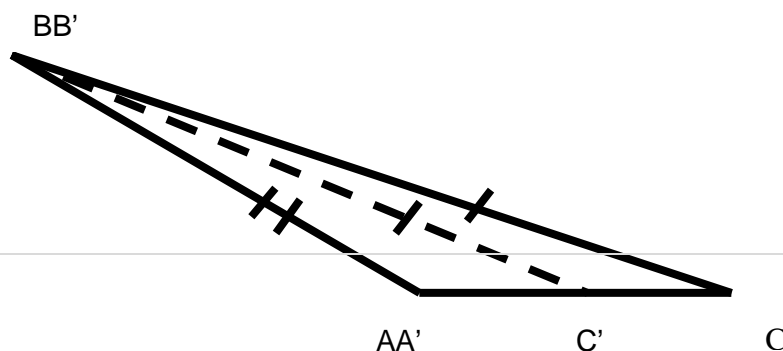
Picture 11. Triangles after matching in the Statement 4.

2. Side $\overline{A'C'}$ goes along side $\overline{AC}$

3. Side $\overline{A'B'}$ goes along side $\overline{AB}$ and coincides with $\overline{A'B'}$ because m∠A = : m∠A' and $\overline{AB} = \overline{A'B'}$.

4. Sides $\overline{BC}$ and $\overline{B'C'}$ make an isosceles triangle ΔC'BB'C because $\overline{AB} = \overline{A'B'}$, and $\overline{BC} = \overline{B'C'}$. In such case ∠CC'BB has to be acute. But this is impossible because ∠C' is acute, and angles ∠A'C'B' and ∠CC'BB' are supplementary. So, in the isosceles triangle ΔC'BB'C one base angle is acute and the other one is obtuse which is impossible.

So, triangles ΔABC and ΔA'B'C' must coincide in all sides, so they are congruent.

End of the proof.

The generalization of four above statements can be summarized in the following

Theorem: If triangles satisfy the SSA conditions, and other angles are both acute or are both obtuse, then such triangles are congruent.

This theorem requires comparing three congruent parts of the triangle (two sides and one not included angle) and one included angle with no exact measure. Such conditions are still weaker than the comparison of measures of six parts of the triangle (three sides and three angles).

It is worthwhile to mention some other statements (theorems) about congruent triangles. Such statements prove that triangles are congruent if they:

a) have congruent two sides and medians to the third side
b) have two sides and congruent altitudes
c) have congruent one side, an altitude, and a bisector.

Such statements are not regularly studied in Geometry courses, but they are available (Atanasian et all., 2015).

2. Conclusions

The previous comparison of Geometry curriculum and four statements proven above allow drawing the following conclusions:

1. Introduction to Geometry should be made in the 8th grade of middle school. At least, it could be done in middle early colleges or magnet schools. The proposed change will allow high school Geometry course to be given at higher level. This will also better prepare students for college tests (SAT or CAT).

2. Statements proven above will allow to include one more theorem about congruent triangles in the Geometry curriculum:

   If two triangles satisfy the SSA conditions, and other pairs of corresponding angles are acute or obtuse, then such triangles are congruent.

   Such theorem can be used as an additional criterion for finding congruent triangles along with other traditional theorems – SAS, ASA, AAS, SSS, and HL.

3. In a high school Math education such theorem requires a review of validity of the use SSA statements in the contemporary Geometry textbooks. Answers to problems related to this statement should be re-evaluated.

4. Contemporary measuring devices, computation methods make the comparison of all sides and angles of triangles in real life very accessible without the application of congruency theorems. In such case the proven statement (theorem) has a restricted value.

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